Numerical methods for wave/flow computations
Part II

Mr. Francesco Lalli

APAT

Agency for Environmental Protection and Technical Services

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1. Remarks on numerical models for small scale coastal problems

Numerical models for small scale coastal problems deal with:

1. wave propagation and interaction with bottom (refraction) and marine structures (diffraction);
2. flows enhanced by wind, waves, large scale circulation.

Generally, the scale ranges from 100m to 1km, and rather fine resolution is required. In fact, waves must be discretized at least by means of 10 nodes per wavelength.

The choice of suitable boundary conditions is also an important aspect.

2. Short waves: Boundary Element Method

As previously stated, the irrotational flow formulation can be a rather general tool for wave-structure interaction:

$$\nabla^2\phi = 0$$

inside the water body

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}$$

at the free surface

$$\frac{\partial \phi}{\partial t} = -\left(g\eta + \frac{p_0}{\rho} \right)$$

at the free surface (typically, in small scale problems p_0=const.)

$$\frac{\partial \phi}{\partial n} = 0$$

at rigid boundaries

2. Short waves: Boundary Element Method

In this method the velocity potential is expressed by means of an integral formulation:

$$\phi(P,t) = \oiint_{\partial D} \frac{\sigma(Q,t)}{|P-Q|} dS_Q$$

3D domains

$$\phi(P,t) = \oint_{\partial D} \sigma(Q,t) \log|P-Q| dS_Q$$

2D domains

P: field point in the water body *D* ; Q: source point at the boundary ∂D

Integral formulation features:

- problem dimensions reduced by 1
- Wave-structure interaction very accurately implemented

2. Short waves: Boundary Element Method

2.1 Numerical wave maker/absorber

$$\frac{\partial \phi}{\partial t} + g\eta = -f'\phi + f\phi_0$$

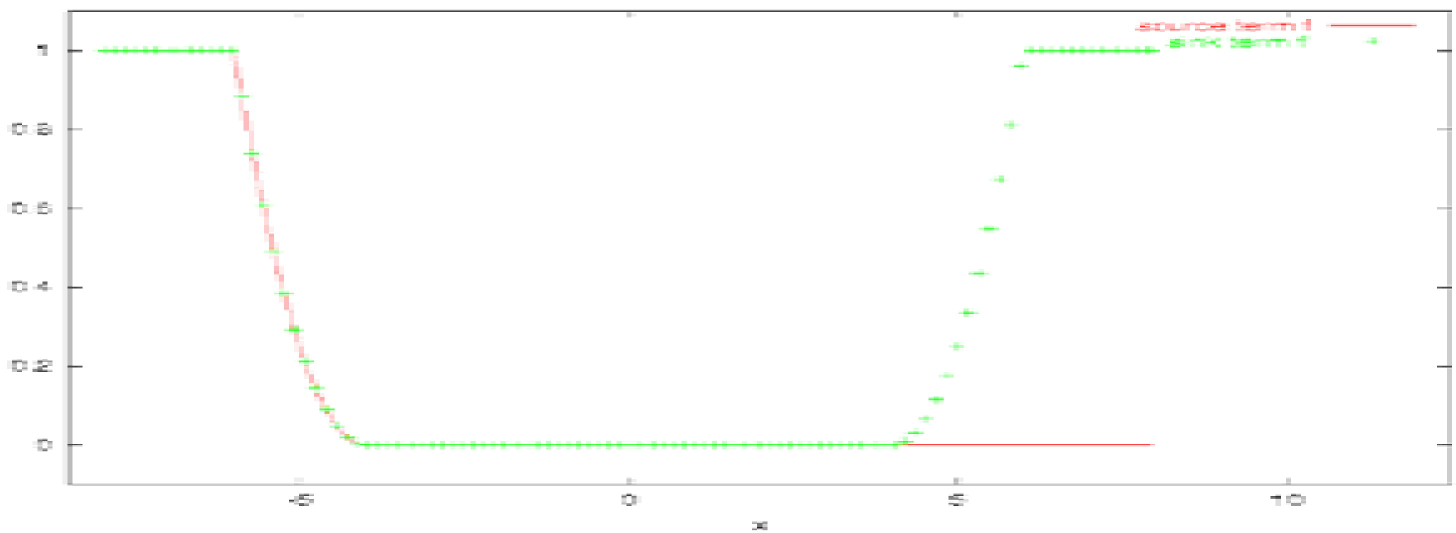
$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = -f'\eta + f\eta_0$$

sink term ***source term***

e. g.

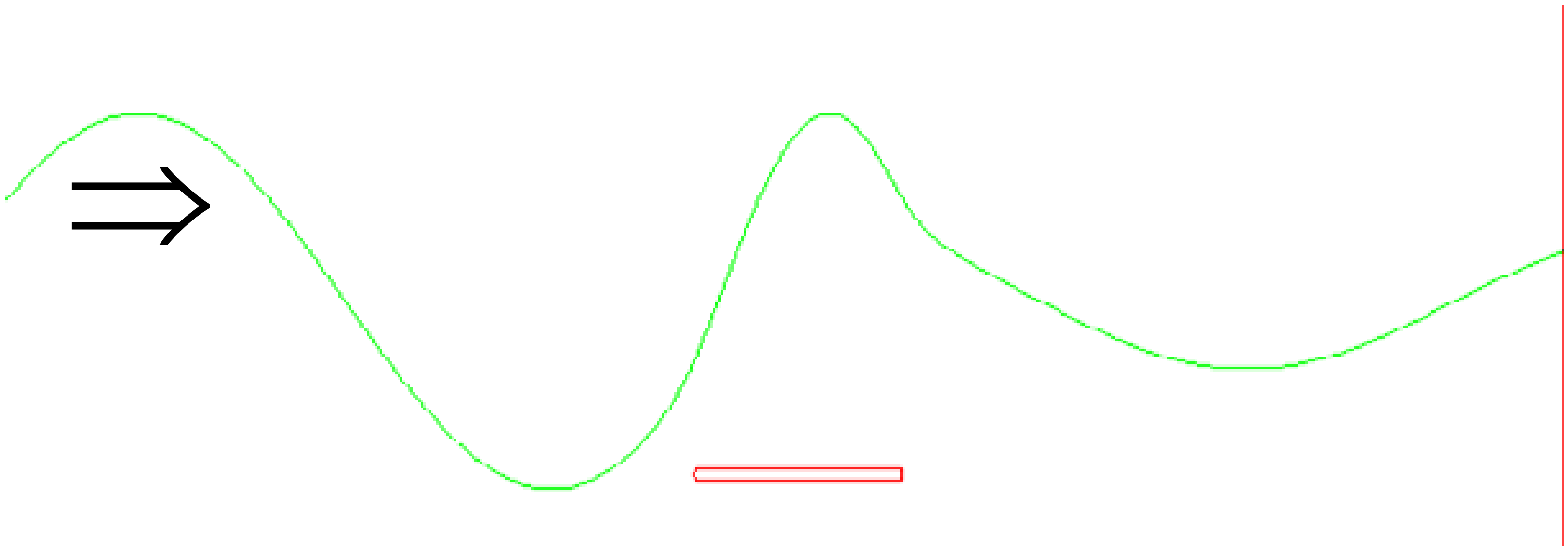
$$\phi_0(x,t) = \frac{gH}{2\omega}\sin\left(kx - \omega t\right)$$

$$\eta_0(x,t) = \frac{H}{2}\cos(kx - \omega t)$$



2. Short waves: Boundary Element Method

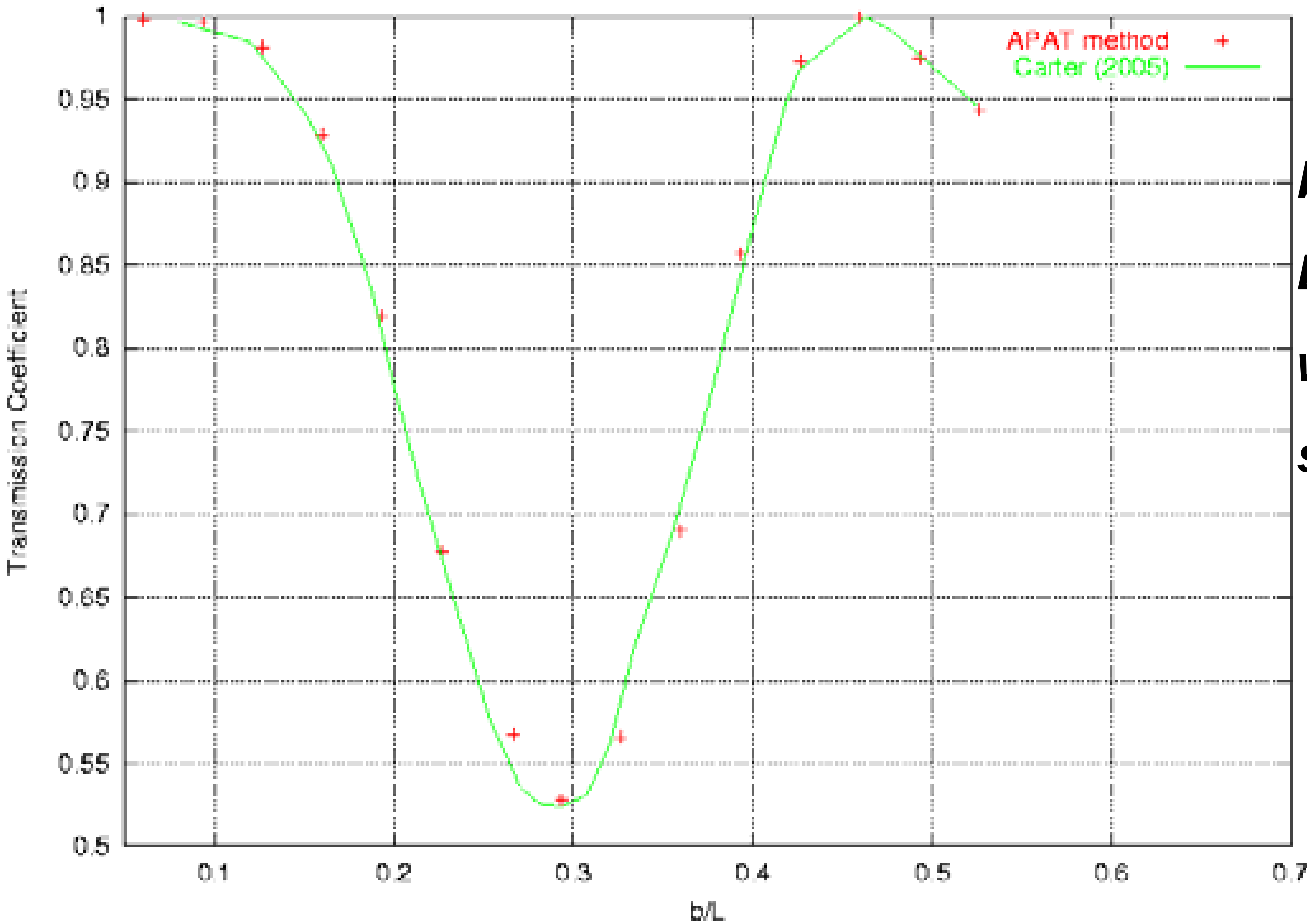
2.2 Application: wave-horizontal flat plate interaction



Attenuation of the wave height due to partial reflection

2. Short waves: Boundary Element Method

2.2 Application: wave-horizontal flat plate interaction



b=plate length

L= wave length

water depth=30cm

submergence=6cm

2. Short waves: Boundary Element Method

2.3 Application: wave-current interaction

$$\frac{\partial \phi}{\partial t} + U_0 \frac{\partial \eta}{\partial x} + g\eta = 0$$

$$\frac{\partial \eta}{\partial t} + U_0 \frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial z}$$

linear waves – uniform stream U_0

$$\frac{A}{A_0} = \left(1 - \frac{U_0 k}{\omega}\right)$$

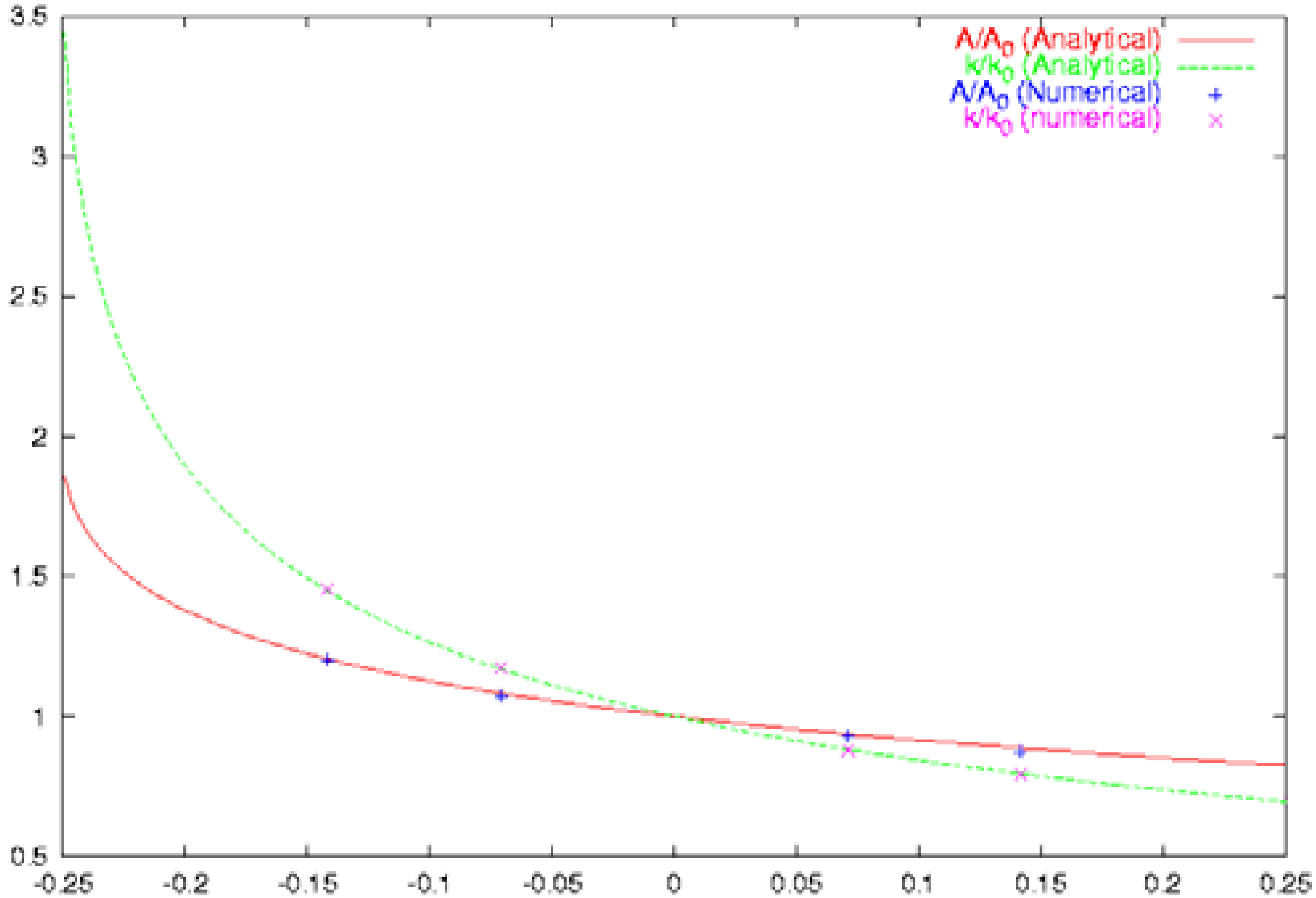
$$\omega = U_0 k + \sqrt{gk \tanh(kh)}$$

amplitude modulation

wave length modulation

2. Short waves: Boundary Element Method

2.3 Application: wave-current interaction



U_0k/ω

3. Long waves: Finite Difference Method

$$\begin{cases}
\dfrac{\partial \eta}{\partial t}+\dfrac{\partial U}{\partial x}+\dfrac{\partial V}{\partial y}=0 \\
\dfrac{\partial U}{\partial t}+\dfrac{\partial\left(U^2/H\right)}{\partial x}+\dfrac{\partial\left(UV/H\right)}{\partial y}=-gH\dfrac{\partial \eta}{\partial x}+\dfrac{2}{\rho}\dfrac{\partial}{\partial x}\left(\mu H\dfrac{\partial\left(U/H\right)}{\partial x}\right)+\dfrac{1}{\rho}\dfrac{\partial}{\partial y}\left[\mu H\left(\dfrac{\partial\left(V/H\right)}{\partial x}+\dfrac{\partial\left(U/H\right)}{\partial y}\right)\right]+\rho\dfrac{gn^2}{H^{7/3}}U\sqrt{U^2+V^2} \\
\dfrac{\partial V}{\partial t}+\dfrac{\partial\left(UV/H\right)}{\partial x}+\dfrac{\partial\left(V^2/H\right)}{\partial y}=-gH\dfrac{\partial \eta}{\partial y}+\dfrac{2}{\rho}\dfrac{\partial}{\partial y}\left(\mu H\dfrac{\partial\left(V/H\right)}{\partial y}\right)+\dfrac{1}{\rho}\dfrac{\partial}{\partial x}\left[\mu H\left(\dfrac{\partial\left(V/H\right)}{\partial x}+\dfrac{\partial\left(U/H\right)}{\partial y}\right)\right]+\rho\dfrac{gn^2}{H^{7/3}}V\sqrt{U^2+V^2}
\end{cases}$$

$$\mu = \nu_0 + \nu_T$$

$$\nu_T = \alpha u^* h$$

eddy viscosity

3. Long waves: Finite Difference Method

3.1 Discrete formulation

$$
\begin{cases}
\eta^{k+1} = \eta^k + \Delta t\left[\gamma A_\eta^k + \rho A_\eta^{k-1}\right] \\
U^{k+1} = U^k + \Delta t\left[\gamma A_U^k + \rho A_U^{k-1}\right] - \alpha \Delta t\left[gH \dfrac{\delta \eta^{k+1}}{\delta x} + FU^{k+1}\right] \\
V^{k+1} = V^k + \Delta t\left[\gamma A_V^k + \rho A_V^{k-1}\right] - \alpha \Delta t\left[gH \dfrac{\delta \eta^{k+1}}{\delta y} + FV^{k+1}\right]
\end{cases}
$$

Runge-Kutta time marching

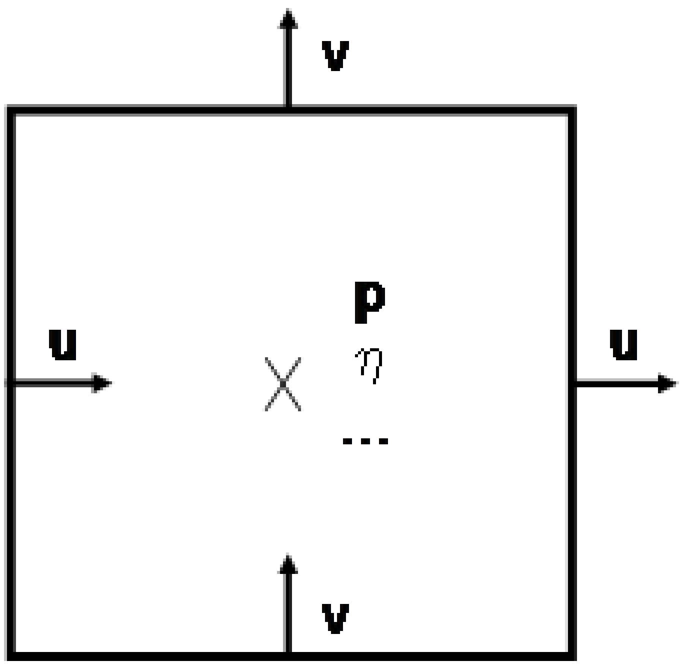
$$A_\eta^k = -\frac{\delta U^k}{\delta x} - \frac{\delta V^k}{\delta y}$$

$$A_U^k = -\frac{\delta}{\delta x}\left[\frac{\left(U^k\right)^2}{H}\right] - \frac{\delta}{\delta y}\left[\frac{U^k V^k}{H}\right] + 2\frac{\delta}{\delta x}\left[\mu \frac{\delta U^k}{\delta x}\right] + \frac{\delta}{\delta y}\left[\mu\left(\frac{\delta V^k}{\delta x} + \frac{\delta U^k}{\delta y}\right)\right]$$

$$A_V^k = -\frac{\delta}{\delta x}\left[\frac{U^k V^k}{H}\right] - \frac{\delta}{\delta y}\left[\frac{\left(V^k\right)^2}{H}\right] + \frac{\delta}{\delta x}\left[\mu\left(\frac{\delta V^k}{\delta x} + \frac{\delta U^k}{\delta y}\right)\right] + 2\frac{\delta}{\delta y}\left[\mu \frac{\delta V^k}{\delta y}\right]$$

3. Long waves: Finite Difference Method

3.1 Discrete formulation



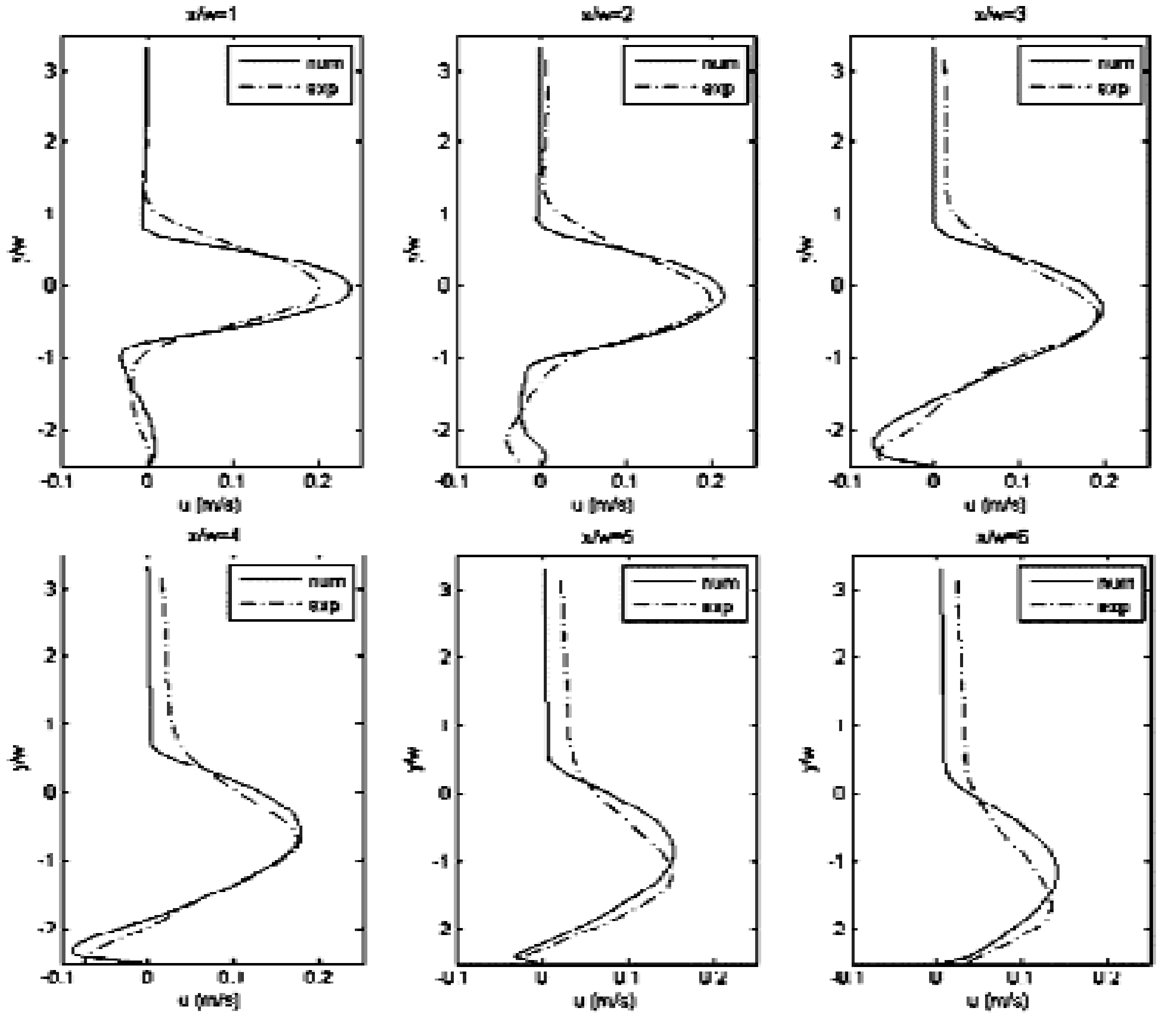
Computational grid

$$\frac{\delta}{\delta x}\left(\frac{U^2}{H}\right) = \frac{\dfrac{U_{i+1}^2}{H_{i+1}} - \dfrac{U_{i-1}^2}{H_{i-1}}}{2\Delta x}$$

Finite Difference scheme

3. Long waves: Finite Difference Method

3.2 Jet-wall interaction



Numerical solutions vs

Experimental data