

General analytical solutions of the linearized Richards equation for a half-space and a finite-thickness domain

Soluzioni analitiche generali dell'equazione di Richards linearizzata per un semispazio e un dominio di spessore finito

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ABSTRACT – This paper aims to describe a compilation of solutions of the linearized one-dimensional Richards equation, solved both in a half-space and in a finite-thickness domain. The solution (the soil water content at any time and depth) can be represented as the sum of two components, one related to the initial condition and to null boundary conditions and the other related to the boundary conditions and to a null initial condition. The sum of the two quoted components is the general solution of the Richards equation in integral form; the analytical expression of the soil water content distribution is therefore obtained if the integrals in the solution can be solved. Besides the integral form solution, another solution holding for any initial and boundary conditions represented with step functions is described in the paper. The initial condition is always the soil water content profile (e.g. the one experimentally measured) while the boundary conditions are different for the two domains. For the half-space domain, the boundary condition can be the soil water content at the surface or the surface flux (e.g. the measured precipitation or evaporation). For this domain, the solution, with the initial-boundary conditions expressed as step functions, is obtained using a procedure, which accounts for the effects of the hydrological conditions of the soil on the flux at the surface. Therefore, this procedure is able to switch between successive atmosphere-controlled and soil-controlled phases of infiltration or evaporation, as required by the given boundary condition. The procedure provides the ponding time, the desiccation time and the surface water flux during the soil-controlled phases. For the finite-thickness domain, the top and bottom boundary conditions are given as time dependent soil water content trends. Also for the finite-thickness domain, a solution, obtained approximating the initial-boundary conditions with step functions, is derived using the basic solution. It provides the soil water content profile evolution, the top and bottom instantaneous and cumulative fluxes and the water gained by the soil layer in a specified time interval. Lastly, a comparison between the procedure results and an exact analytical solution is discussed.

KEY WORDS: homogeneous materials, mathematical models, moisture, one-dimensional models, porous materials, surficial aquifers, theoretical models, unsaturated zone.

RIASSUNTO – Questo lavoro è un sommario di soluzioni dell'equazione unidimensionale linearizzata di Richards per un semispazio ed un dominio di spessore finito. La soluzione cercata è il contenuto volumetrico d'acqua del suolo. Essa è la somma di due componenti: la soluzione ottenuta con condizione iniziale assegnata e con condizione al contorno nulla e la soluzione ottenuta con condizione iniziale nulla e condizione al bordo assegnata. La forma integrale della soluzione così ottenuta fornisce l'espressione analitica del contenuto volumetrico d'acqua se gli integrali che contiene sono risolvibili. Il lavoro, oltre a presentare la soluzione generale in forma integrale, fornisce le soluzioni per qualsiasi condizione iniziale e al contorno, approssimate con funzioni a gradini. La condizione iniziale è, per entrambi i domini, il profilo verticale di contenuto volumetrico d'acqua del suolo. La condizione al contorno, per il semispazio, può essere sia il flusso sia il contenuto volumetrico d'acqua alla superficie; per lo strato di spessore finito, le condizioni su entrambi i bordi sono sempre il contenuto volumetrico d'acqua. Il profilo verticale del contenuto volumetrico d'acqua, misurato in una stazione idrometeorologica, è già una funzione a gradini perfettamente utilizzabile come condizione iniziale, analogamente il flusso d'acqua misurato all'interfaccia aria-suolo (precipitazione o evaporazione) è, a sua volta, una funzione a gradini immediatamente utilizzabile come condizione al bordo per la soluzione del problema. Per questo dominio (semispazio) è stata messa a punto una procedura che tiene conto degli effetti delle condizioni idrologiche del suolo sul flusso alla superficie. La procedura è in grado di gestire automaticamente il passaggio da una fase di evaporazione o infiltrazione controllata dall'atmosfera ad una controllata dal suolo e viceversa, come richiesto dalla condizione al contorno assegnata. La procedura fornisce anche il tempo al quale il suolo non è più

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in grado di assorbire o cedere l'acqua fornita o richiesta dall'atmosfera; inoltre calcola i valori istantanei e cumulativi del flusso alla superficie, del ruscellamento, dell'acqua guadagnata da una colonna di suolo e del flusso uscente dal fondo di tale colonna. Viene inoltre mostrato e discusso un confronto tra i risultati di questa procedura e una soluzione analitica. Infine, anche per lo strato di spessore finito, è presentata una soluzione ottenuta approssimando le condizioni iniziali e ai bordi con funzioni a gradini e vengono calcolati i valori istantanei e cumulativi del flusso al bordo superiore e inferiore e dell'acqua guadagnata dalla colonna di suolo.

PAROLE CHIAVE: acquifero superficiale, mezzo omogeneo, mezzo poroso, modello matematico, modello monodimensionale, modello teorico, umidità del suolo, zona non satura.

1. – INTRODUCTION

The space and time evolution of the soil water content in an unsaturated medium is described by the Richards equation (RICHARDS, 1931). This equation is highly non linear because of the dependence of both the hydraulic conductivity and the soil water potential on the soil water content. Therefore, several numerical routines have been developed to solve the Richards equation with numerical schemes. However, it is well recognized that analytical solutions of differential equations describing physical problems provide general insights and concisely identify the relationships among the variables of the studied problems, allowing rational approximations and simplifications. Therefore, although numerical methods are powerful in solving complex non-linear problems, analytical solutions maintain their utility and can also provide a useful check to numerical procedures. Some particular cases, exact and approximated analytical solutions of the Richards equation have been derived by, e.g., SANDERS *et alii* (1988), HOGARTH *et alii* (1989), HOGARTH *et alii* (1992), PARLANGE *et alii* (1992), ROSS & PARLANGE (1994), PARLANGE *et alii* (1997), HOGARTH & PARLANGE (2000). Moreover, analytical solutions of the linearized Richards equation have been derived in integral form by WARRICK (1975) and BASHA (1999) resulting in closed form solutions only for constant flux boundary conditions. CHEN *et alii* (2001) derived analytical solutions of the linearized Richards equation for a variety of time dependent fluxes, before surface saturation, while CHEN *et alii* (2003) presented a linearized solution technique for a variety of surface fluxes after ponding. To obtain limiting cases of the real soil solutions, WANG & DOOGE (1994) proposed some analytical solutions of the Richards equation using a uniform initial condition and a quantized flux boundary condition. Two different approaches were used by MENZIANI *et alii* (2005) to obtain exact solutions of the

Richards equation: one is used to solve the non-linear equation without the gravity term, the other allows to derive solutions to the linearized equation. A hybrid procedure, making use of initial-boundary conditions approximated by step functions, has been carried out by MENZIANI *et alii* (2007).

In the present work, a compilation of solutions of the linearized one-dimensional Richards equation is presented. This work is based on two spatial schemes: a half-space and a finite-thickness domain. The computed soil water content at any time and depth is always the sum of two components. One component is related to the initial condition and one or two null boundary conditions; the other is related to the boundary conditions and to a null initial condition. The sum of the two components gives the general solution of the Richards equation in integral form. This integral form is not always solvable; here some exact solutions are given.

Finally, another solution, holding for any initial and boundary conditions represented with step functions, is described in the paper.

2. – MATHEMATICAL FORMULATION

In this paragraph, the mathematical formulation to solve the onedimensional linearized flow equation to study infiltration and evaporation processes in a homogeneous medium is presented. A half-space and a finite-thickness domain are the two schemes used to model a nondeformable soil. For both schemes, the linearized Richards equation is changed in a new equation containing only dimensionless variables, suitable for general application. For the finite-thickness domain scheme only this first part will be developed while more detailed results will be given for the half-space domain.

Assuming t as the time, z as the soil depth (positive downward) D as the constant hydraulic diffusivity and $V = \partial k / \partial \theta$ as the first derivative of the hydraulic conductivity ($k(\theta)$), the following equation

$$\frac{\partial Y}{\partial t} = D \cdot \frac{\partial^2 Y}{\partial z^2} - V \cdot \frac{\partial Y}{\partial z} \quad (1)$$

can be used to describe the space and time evolution of the volumetric soil water content,

$$\theta(z, t) \quad (0 \leq \theta \leq 1)$$

or to describe the evolution of the corresponding flux, defined as:

$$\Phi(z, t) = V \cdot \theta - D \cdot \frac{\partial \theta}{\partial z} \quad (2)$$

In order to use the linear differential equation (1), the relationship between the hydraulic conductivity and the water content is assumed to be linear. The problem to determine $\mathcal{G}(z,t)$ is an initial condition problem (except for periodic boundary conditions) therefore an initial condition is required:

$$\mathcal{G}(z,0) = \mathcal{G}_i(z) \quad (t=0) \quad (3a)$$

or:

$$\Phi(z,0) = \Phi_i(z) \quad (t=0) \quad (3b)$$

The space domain is $z \geq 0$ for a half-space or $0 \leq z \leq H$ for a finite-thickness domain of depth H. For the half-space only one boundary condition is given:

$$\mathcal{G}(0,t) = \mathcal{G}_0(t) \quad (t > 0) \quad (4a)$$

or:

$$\Phi(0,t) = \Phi_0(t) \quad (t > 0) \quad (4b)$$

while for the finite-thickness domain two boundary conditions are required:

$$\begin{aligned} \mathcal{G}(0,t) &= \mathcal{G}_0(t) & (t > 0) \\ \mathcal{G}(H,t) &= \mathcal{G}_H(t) & (t > 0) \end{aligned} \quad (5a)$$

$$\begin{aligned} \text{or: } \Phi(0,t) &= \Phi_0(t) & (t > 0) \\ \Phi(H,t) &= \Phi_H(t) & (t > 0) \end{aligned} \quad (5b)$$

From the hydraulic parameters of the medium, a characteristic length $\delta = D/V$ can be defined, which allows to introduce the dimensionless variables:

$$\zeta = z/\delta \quad \eta = D \cdot t/\delta^2$$

Using these two dimensionless variables, equation (1) results:

$$\frac{\partial Y}{\partial \eta} = \frac{\partial^2 Y}{\partial \zeta^2} - \lambda \cdot \frac{\partial Y}{\partial \zeta} \quad (\zeta \geq 0; \eta \geq 0) \quad (6)$$

When treating with the finite-thickness domain, H (the depth of the layer) represents a reference (or scale) length and, in this case, it is convenient to define the dimensionless space and time variables as follows:

$$\zeta = z/H \quad \eta = D \cdot t/H^2$$

and equation (1) becomes:

$$\frac{\partial Y}{\partial \eta} = \frac{\partial^2 Y}{\partial \zeta^2} - \lambda \cdot \frac{\partial Y}{\partial \zeta} \quad (0 \leq \zeta \leq 1; \eta \geq 0); \lambda = (V \cdot H)/D \quad (7)$$

2.1. – HALF-SPACE DOMAIN: PERIODIC BOUNDARY CONDITION

In some hydrological problems, a periodic boundary condition (PBC) is particularly useful to describe the soil volumetric water content or the flux at the surface. The solution, in this case, is easily obtained because this is no more an initial condition problem.

Considering a PBC, with amplitude \mathcal{G}^* and angular frequency ω , for the soil volumetric water content:

$$\mathcal{G}_0(t) = \mathcal{G}^* \cdot e^{i\omega t}$$

and assuming that the method of separation of variables is appropriate, the searched solution can be written as:

$$\mathcal{G}(z,t) = e^{i\omega t} \cdot \mathcal{G}(z)$$

Now, considering the half-space scheme, the solution of equation (6) is:

$$\mathcal{G}(\zeta, \eta) = \mathcal{G}^* \cdot e^{-\left[-1 + \frac{\sqrt{1 + \sqrt{1 + \mu^2}}}{\sqrt{2}} \right] \frac{\zeta}{2}} \cdot e^{i \left[\left(\frac{\delta}{\delta_0} \right)^2 \cdot \eta + \frac{-\mu^2}{\sqrt{2} \cdot \sqrt{1 + \sqrt{1 + \mu^2}}} \frac{\zeta}{2} \right]}$$

Where: $\mu^2 = 8 \cdot \omega \cdot D/V^2$ and $\delta_0^2 = D/\omega$ which is the characteristic depth of the problem for the considered period; δ, ζ, η are defined above. Of course, $\omega = 0$ corresponds to the stationary state and $\mathcal{G}(\zeta, \eta) = \mathcal{G}^*$.

A solution $\Phi(\zeta, \eta)$, similar to $\mathcal{G}(\zeta, \eta)$, can be obtained considering a PBC for the flux at the surface. In this case, to get the soil volumetric water content, the integration of the water flux have to be performed (see the following equation (11)).

2.2. – HALF-SPACE DOMAIN: GENERAL SOLUTION

Here it will be examined the case of a homogeneous half-space. To solve equation (6) an initial condition and one time dependent boundary condition have to be assigned.

Considering for the soil water content any continuous function of the vertical depth as initial condition and any continuous function of the time as boundary condition, the general solution of equation (6), due to its linearity, can be obtained as the sum of two solutions:

$$\mathcal{G}(\zeta, \eta) = \mathcal{G}_1(\zeta, \eta) + \mathcal{G}_2(\zeta, \eta) \quad (8)$$

Where $\mathcal{G}_1(\zeta, \eta)$ represents the solution of equation (6) with the initial condition (3a) and a null boundary condition and $\mathcal{G}_2(\zeta, \eta)$ represents the solution of equation (6) with a null initial condition and the boundary condition (4a). That is:

$$g_1(\zeta, \eta) = \frac{e^{\frac{\zeta+\eta}{2}}}{\sqrt{\pi} \sqrt{4 \cdot \eta}} \cdot \int_0^{\infty} g_1(\zeta') \cdot e^{-\frac{\zeta'}{2}} \left[e^{\frac{(\zeta-\zeta')^2}{4 \cdot \eta}} - e^{\frac{(\zeta+\zeta')^2}{4 \cdot \eta}} \right] \cdot d\zeta' \quad (9a)$$

and, by means of the Duhamel theorem (CARSLAW & JAEGER, 1986), one obtains:

$$g_2(\zeta, \eta) = \frac{2}{\sqrt{\pi}} \cdot \int_0^{\eta} g_0(\eta') \cdot e^{-\frac{(\eta-\eta')-\zeta}{\sqrt{4 \cdot (\eta-\eta')}}} \cdot \frac{4 \cdot \zeta \cdot d\eta'}{2 \cdot [4 \cdot (\eta-\eta')]^{\frac{3}{2}}} \quad (9b)$$

The sum of equations (9a) and (9b) is the integral form of the solution of equation (6) but the integrals in the aforementioned equations can be difficult or even impossible to be solved analytically.

Similarly, if the flux is the unknown function in equation (6), the general solution can be obtained as the sum of two solutions:

$$\Phi(\zeta, \eta) = \Phi_1(\zeta, \eta) + \Phi_2(\zeta, \eta) \quad (10)$$

Where $\Phi_1(\zeta, \eta)$ represents the solution of the flux-based form of equation (6) with the initial condition (3b) and a null boundary condition and $\Phi_2(\zeta, \eta)$ represents the solution of the flux-based form of equation (6) with a null initial condition and the boundary condition (4b). Their expressions are equal to (9a) and (9b) with Φ instead of g .

Once the solution $\Phi(\zeta, \eta)$ has been computed, the soil volumetric water content $g(\zeta, \eta)$ is obtained from equation (2):

$$g(\zeta, \eta) = \frac{e^{\zeta}}{V} \cdot \int_{\zeta}^{\infty} e^{-\zeta'} \cdot \Phi(\zeta', \eta) \cdot d\zeta' \quad (11)$$

remembering that: $\lim_{\zeta \rightarrow \infty} e^{-\zeta} \cdot g(\zeta, \eta) = 0$.

2.3. – FINITE-THICKNESS DOMAIN: GENERAL SOLUTION

Let us consider now the case of a homogeneous H-thick finite-thickness domain. To solve equation (7) an initial condition and two time dependent boundary conditions have to be assigned. As for the previous domain, the general solution of equation (7) is given by the sum of two solutions as described by equation (8). Again, the solution $g_1(\zeta, \eta)$ is derived for null boundary conditions and an arbitrary initial condition (3a) and $g_2(\zeta, \eta)$ is derived for a null initial condition and two (usually different) arbitrary boundary conditions (5a). That is:

$$g_1(\zeta, \eta) = e^{\frac{\lambda \cdot \zeta}{2}} \cdot \sum_{n=1}^{\infty} B_n \cdot e^{-\Lambda_n \cdot \eta} \cdot \sin(n \cdot \pi \cdot \zeta) \quad (12a)$$

$$\text{with } \Lambda_n = \left(n^2 \cdot \pi^2 + \frac{\lambda^2}{4} \right)$$

where:

$$B_n = 2 \cdot \int_0^1 g_1(\zeta) \cdot e^{-\frac{\lambda \cdot \zeta}{2}} \cdot \sin(n \cdot \pi \cdot \zeta) \cdot d\zeta$$

Also in this case, for a null initial condition and two continuous time functions as boundary conditions, the general analytical solution of equation (7) is obtained using the Duhamel theorem.

$$g_2(\zeta, \eta) = e^{-\frac{\lambda \cdot \zeta}{2}} \sum_{n=1}^{\infty} \left\{ 2n\pi \cdot e^{-\Lambda_n \cdot \eta} \sin(n\pi\zeta) \cdot \left[\int_0^{\eta} g_0(\eta') \cdot e^{\Lambda_n \cdot \eta'} \cdot d\eta' - (-1)^n \cdot e^{-\frac{\lambda \cdot \eta}{2}} \int_0^{\eta} g_n(\eta') \cdot e^{\Lambda_n \cdot \eta'} \cdot d\eta' \right] \right\} \quad (12b)$$

The sum of equations (12a) and (12b) is the solution of equation (7) in integral form. The analytical expression of the soil water content distribution $g(\zeta, \eta)$ is obtained if the integrals in the above equations can be solved.

3. – HALF-SPACE DOMAIN: PROCEDURE

In this section, a procedure to compute the half-space solution, obtained approximating the initial-boundary conditions with step functions, is described. First of all, let us present the solution obtained assuming a null initial condition and a constant boundary condition.

3.1. – HALF-SPACE DOMAIN: NULL IC AND CONSTANT BC

Let us assume the following initial-boundary conditions:

$$\begin{aligned} g_1(\zeta) &= 0 & (\eta = 0) \\ g_0(\eta) &= g & (\zeta = 0) \end{aligned}$$

Due to the choice of the null initial condition the component (9a) of the solution is null, therefore the problem reduces to the determination of (9b)

$$g(\zeta, \eta) = \frac{g}{2} \cdot \left[\operatorname{erfc} \left(\frac{\zeta - \eta}{2 \cdot \sqrt{\eta}} \right) + e^{\zeta} \cdot \operatorname{erfc} \left(\frac{\zeta + \eta}{2 \cdot \sqrt{\eta}} \right) \right] \quad (13)$$

Using the same initial-boundary conditions for the water flux a similar solution is found with Φ instead of g .

3.2. – PROCEDURE

This procedure computes the solution of the linearized Richards equation, approximating the initial-boundary conditions with step functions.

The initial condition is the soil volumetric water content, while the boundary condition is the surface flux. In fact, the soil volumetric water content profile and the precipitation or evaporation are usually measured at hydro-meteorological stations.

As explained, the solution is the sum of $g(\zeta, \eta)$, that is the component obtained assuming a null boundary condition and an arbitrary initial condi-

tion, and $\vartheta_2(\zeta, \eta)$, that is the component obtained assuming a null initial condition and an arbitrary boundary condition. The procedure changes the boundary condition from the water flux to the soil water content and vice versa, according to the atmosphere-controlled or soil-controlled phase respectively.

3.2.1. – null BC and arbitrary IC

Let us assume an arbitrary initial condition approximated by:

$$\vartheta_1(\zeta) = \vartheta_0 + \sum_{n=1}^N (\vartheta_n - \vartheta_{n-1}) \cdot H(\zeta - \zeta_n)$$

N is the total number of discontinuities (at: $\zeta_1, \zeta_2, \dots, \zeta_N$) where the initial condition assumes the values: $\vartheta_1, \vartheta_2, \dots, \vartheta_N$ (besides ϑ_0 , which is the soil water content value between $\zeta = 0$ and $\zeta = \zeta_1$).

$H(\zeta - \zeta_n)$ is the Heaviside function (JONES, 1966).

The last soil water content value ϑ_N is constant between $\zeta = \zeta_N$ and infinity. With a null flux as boundary condition and taking into account the principle of superposition, the solution is given by the sum of $N+1$ terms.

$$\begin{aligned} \vartheta_1(\zeta, \eta) = & \frac{\vartheta_0}{2} \left[\operatorname{erfc} \left(\frac{\eta - \zeta}{2 \cdot \sqrt{\eta}} \right) + e^\zeta \cdot \left[\operatorname{erfc} \left(\frac{\eta + \zeta}{2 \cdot \sqrt{\eta}} \right) - 2 \cdot \sqrt{\eta} \cdot \operatorname{ierfc} \left(\frac{\eta + \zeta}{2 \cdot \sqrt{\eta}} \right) \right] \right] + \\ & + \sum_{n=1}^N \frac{(\vartheta_n - \vartheta_{n-1})}{2} \cdot \left[\operatorname{erfc} \left(\frac{\eta + \zeta_n - \zeta}{2 \cdot \sqrt{\eta}} \right) + e^\zeta \cdot \left[\operatorname{erfc} \left(\frac{\eta + \zeta_n + \zeta}{2 \cdot \sqrt{\eta}} \right) - 2 \cdot \sqrt{\eta} \cdot \operatorname{ierfc} \left(\frac{\eta + \zeta_n + \zeta}{2 \cdot \sqrt{\eta}} \right) \right] \right] \end{aligned} \quad (14)$$

3.2.2. – null IC and arbitrary BC

3.2.2.1. – atmosphere-controlled phase

During the pre-ponding phase or pre-desiccation-phase the rate of the flux at the surface is atmosphere-controlled. Supposing to be in this phase, let us assume an arbitrary flux boundary condition approximated with a step function:

$$\Phi_0(\eta) = q_0 + \sum_{m=1}^M (q_m - q_{m-1}) \cdot H(\eta - \eta_m)$$

M is the total number of discontinuities (at: $\eta_1, \eta_2, \dots, \eta_M$) where the boundary condition assumes the values: q_1, q_2, \dots, q_M (besides q_0 , which is the surface flux value between $\eta = 0$ and $\eta = \eta_1$). Assuming a uniform initial condition $\vartheta_1(\zeta) = 0$ and taking into account the principle of superposition, the solution of the problem will be a linear combination of $(J_\eta + 1)$ terms, with $J_\eta \leq M$. J_η is the number of the discontinuities before η ; e.g. if $\eta_2 < \eta < \eta_3$ then $J_\eta = 2$.

$$\begin{aligned} \vartheta_2(\zeta, \eta) = & \frac{q_0}{2 \cdot V} \left[1 + \operatorname{erf} \left(\frac{\eta - \zeta}{2 \cdot \sqrt{\eta}} \right) + e^\zeta \cdot \left[2 \cdot \sqrt{\eta} \cdot \operatorname{ierfc} \left(\frac{\eta + \zeta}{2 \cdot \sqrt{\eta}} \right) - \operatorname{erfc} \left(\frac{\eta + \zeta}{2 \cdot \sqrt{\eta}} \right) \right] \right] + \sum_{m=1}^{J_\eta} \frac{(q_m - q_{m-1})}{2 \cdot V} \cdot \\ & \left[1 + \operatorname{erf} \left(\frac{\eta - \eta_{m-1} - \zeta}{2 \cdot \sqrt{\eta - \eta_{m-1}}} \right) + e^\zeta \cdot \left[2 \cdot \sqrt{\eta - \eta_{m-1}} \cdot \operatorname{ierfc} \left(\frac{\eta - \eta_{m-1} + \zeta}{2 \cdot \sqrt{\eta - \eta_{m-1}}} \right) - \operatorname{erfc} \left(\frac{\eta - \eta_{m-1} + \zeta}{2 \cdot \sqrt{\eta - \eta_{m-1}}} \right) \right] \right] \end{aligned} \quad (15)$$

The solution of the linearized Richards equation is given by the sum of equation (14) and (15) during the atmosphere-controlled phase.

3.2.2.2. – soil-controlled phase

If the precipitation rate exceeds the soil hydraulic conductivity at saturation, the downward infiltration rate into an initially unsaturated soil corresponds to the rainfall rate until the ponding is reached. Thereafter, the rate of infiltration will depend on the soil hydraulic characteristics and the phase is called “soil-controlled phase”. Similarly, during drying periods, the evaporation mechanism switches from atmosphere to soil controlled phase when the soil is no longer able to supply water at the rate required by the atmosphere. At the beginning of the soil-controlled phase a new initial condition is required. It is the discrete soil moisture vertical profile computed during the atmosphere-controlled phase at the ponding time η_p (saturation) or at the desiccation time η_d (air-dry soil, when the rate of the water loss at the surface exceeds the rate of supply from below). The boundary condition is thereafter a soil water content constant value equal to one in case of ponding and equal to zero in case of air-dry soil.

3.2.2.3. – desiccation

If the air-dry soil is reached, a null soil volumetric water content at the surface is the new boundary condition and the new initial condition is the vertical profile of the atmosphere-controlled phase at $\eta = \eta_d$.

$$\begin{aligned} \vartheta_d(\zeta, \eta - \eta_d) = & \sum_{n=0}^N \frac{\vartheta_n}{2} \cdot \left[\operatorname{erfc} \left(\frac{\eta - \eta_d + \zeta_n - \zeta}{2 \cdot \sqrt{\eta - \eta_d}} \right) - \operatorname{erfc} \left(\frac{\eta - \eta_d + \zeta_{n+1} - \zeta}{2 \cdot \sqrt{\eta - \eta_d}} \right) - \right. \\ & \left. e^\zeta \cdot \left[\operatorname{erfc} \left(\frac{\eta - \eta_d + \zeta_n + \zeta}{2 \cdot \sqrt{\eta - \eta_d}} \right) - \operatorname{erfc} \left(\frac{\eta - \eta_d + \zeta_{n+1} + \zeta}{2 \cdot \sqrt{\eta - \eta_d}} \right) \right] \right] \end{aligned} \quad (16)$$

3.2.2.4. – ponding

If the ponding is reached, unitary soil volumetric water content at the surface is the new boundary condition and the new initial condition is the vertical profile of the atmosphere-controlled phase at $\eta = \eta_p$.

$$\vartheta_p(\zeta, \eta - \eta_p) = \vartheta_d(\zeta, \eta - \eta_p) + 1 - \frac{1}{2} \cdot \left[\operatorname{erfc} \left(\frac{\eta - \eta_p - \zeta}{2 \cdot \sqrt{\eta - \eta_p}} \right) - e^\zeta \cdot \operatorname{erfc} \left(\frac{\eta - \eta_p + \zeta}{2 \cdot \sqrt{\eta - \eta_p}} \right) \right] \quad (17)$$

As usual the solution (17) is the sum of two parts, the first is the solution (16) obtained with the aforementioned new initial condition and a null soil

water content boundary condition. In this case, η_p instead of η_d has to be used in equation (16). The second part is the solution for a null initial condition and a unitary soil water content boundary condition.

4. – ILLUSTRATIVE EXAMPLES

In the previous section, the general solution of the linearized Richards equation has been obtained in integral form; here particular analytical solutions pertaining to some simple cases of initial-boundary conditions are deduced from the general analytical model.

In this section it is also shown an example of the procedure, which uses the step functions as initial-boundary conditions. It will show the ability of this procedure to take into account the atmosphere-controlled and soil-controlled phases of infiltration or evaporation, as required by the given boundary condition. A comparison between the result of this procedure and an exact analytical solution is then discussed.

A very simple example for the finite-thickness domain is finally presented. The analytical solution used in this example, which assumes constant initial and boundary conditions, represents the basic element to build a procedure similar to the one described for the half-space domain. The solution with the initial-boundary conditions approximated with step functions is a sum of solutions similar to the one here used. Since the boundary conditions for the finite-thickness domain are the soil water content trends, a switch between atmosphere-controlled and soil-controlled phases is not required.

4.1. – HALF-SPACE DOMAIN: NULL IC AND EXPONENTIAL BC

Let us assume the following initial-boundary conditions.

$$\begin{aligned} \vartheta_1(\zeta) &= 0 & (\eta = 0) \\ \vartheta_0(\eta) &= \vartheta_1 \cdot e^{-\gamma\eta} & (\zeta = 0) \end{aligned}$$

The choice of the null initial condition implies that the component (9a) of the solution is null and the problem reduces to the determination of (9b).

According to the boundary condition parameter γ , the solution has three different expressions. This is due to the presence of a square root with a radicand, which is null for $\gamma=1/4$, positive for $\gamma < 1/4$ and negative for $\gamma > 1/4$. Therefore:

- for $\gamma < 1/4$

$$\vartheta(\zeta, \eta) = \frac{\vartheta_1}{2} e^{\left(\frac{\zeta-2\gamma\eta}{2}\right)} \left[e^{\zeta\sqrt{\frac{1-\gamma}{4}}} \operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\eta}} + \sqrt{\eta}\sqrt{\frac{1-\gamma}{4}}\right) + e^{-\zeta\sqrt{\frac{1-\gamma}{4}}} \operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\eta}} - \sqrt{\eta}\sqrt{\frac{1-\gamma}{4}}\right) \right] \tag{18a}$$

It is easy to verify that, if $\gamma=0$, equation (18a) reduces to equation (13).

- for $\gamma=1/4$

$$\vartheta(\zeta, \eta) = \vartheta_1 \cdot e^{\left(\frac{2\zeta-\eta}{4}\right)} \cdot \operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\eta}}\right) \tag{18b}$$

- for $\gamma > 1/4$

$$\vartheta(\zeta, \eta) = \vartheta_1 \cdot e^{\left(\frac{\zeta-\eta}{2\sqrt{\eta}}\right)^2} \cdot \Re\left\{ W\left(\sqrt{\gamma-\frac{1}{4}} \cdot \sqrt{\eta} + i \frac{\zeta}{2\sqrt{\eta}}\right) \right\} \tag{18c}$$

where $W(x+iy) \equiv W(z) = e^{-z^2} \cdot \operatorname{erfc}(-i \cdot z)$ is a complex function (of complex variable) whose real and imaginary parts are reported in table 7.9, p. 326 (ABRAMOWITZ & STEGUN, 1965) and in the Appendix II, table II, p. 486 (CARSLAW & JAEGER, 1986). Equation (18c) reduces to equation (18b) if $\gamma=1/4$, the same for equation (18a).

4.2. – HALF-SPACE DOMAIN: UNIFORM IC AND CONSTANT POSITIVE AND NEGATIVE FLUX BC

In order to understand how works the above described procedure, a first application is presented. Uniform soil water content is assumed as initial condition; the flux boundary condition is a step function: starting at $\eta=0$ with a positive value; at $\eta=10 \cdot \eta_0$ the flux becomes negative with the same absolute value. Here, considering a typical soil, η_0 is the dimensionless time corresponding to half an hour. The assumed flux is the boundary condition only during the atmosphere-controlled phases. During the soil-controlled phases, the boundary condition becomes the soil water content at saturation ($\vartheta=1$) or at desiccation ($\vartheta=0$). In the presented application, a sequence of two consecutive switches between atmosphere-controlled and soil-controlled phases is solved. The results are for a column of soil of dimensionless thickness $\zeta=3$.

In figure 1a, the thick dashed line corresponds to the uniform soil volumetric water content distribution at the starting time, the initial condition.

The vertical soil moisture distributions at the quoted multiples of η_0 are drawn in blue and red during the atmosphere-controlled phases and in cyan and pink during the soil-controlled phases. With the available positive flux applied, the soil surface reaches the saturation at the ponding time η_p and remains saturated until the applied flux becomes negative. At this time a new atmosphere-controlled phase starts and it lasts till the soil is able to convey water to the surface at the poten-

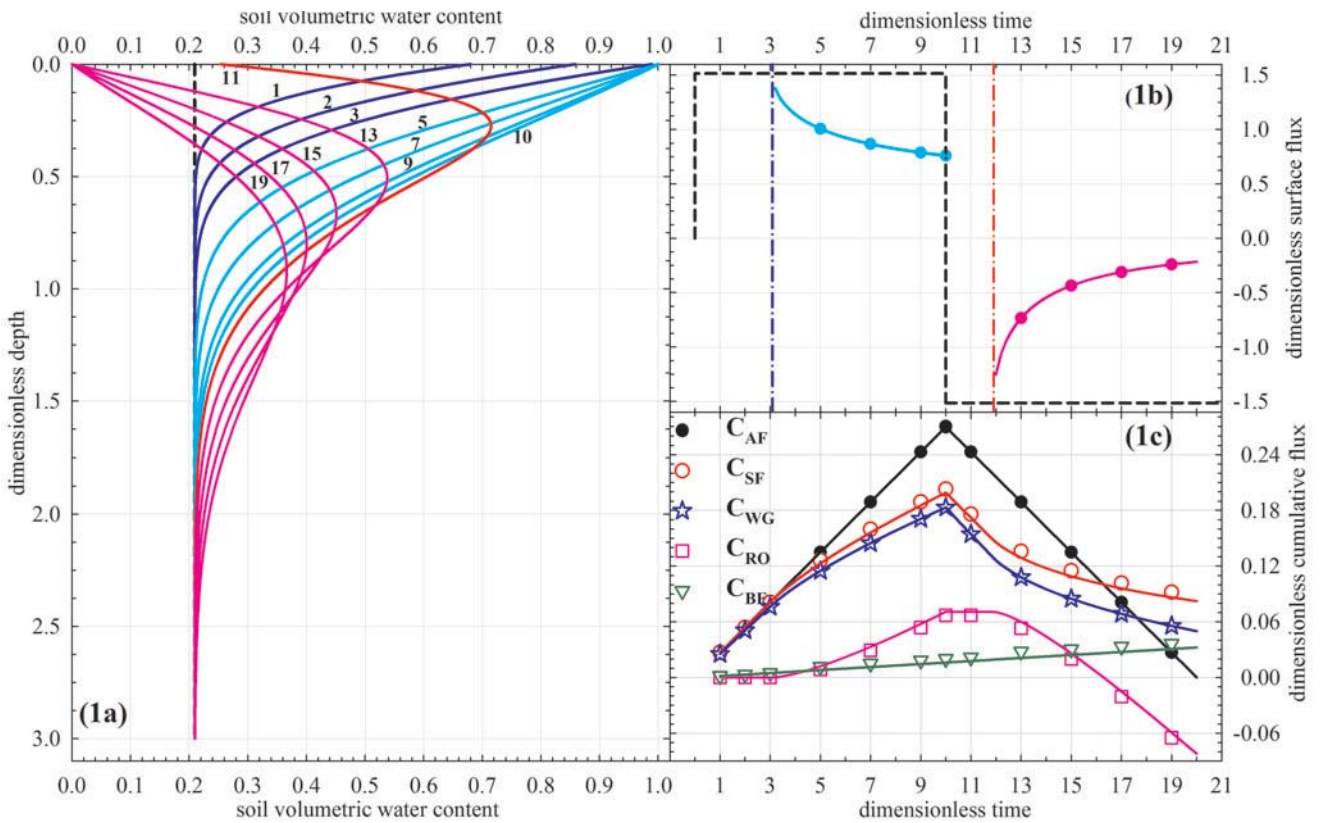


Fig. 1 - (a) soil moisture profiles computed by the procedure at the indicated dimensionless times (multiples of η_0), the blue and red curves are the solutions during the atmosphere-controlled phases and the cyan and pink ones during the soil-controlled phases; (b) applied surface flux (thick dashed line) and analytical surface flux after ponding time (cyan) or after desiccation time (pink), the two thin dash-dot vertical lines indicate the ponding time (blue) and the desiccation time (red); (c) cumulative flux trends computed by the procedure: applied surface flux (C_{AF}), water entered through the surface (C_{SF}), water gained by the column of soil of dimensionless height $\zeta = 3$ (C_{WG}), runoff (positive) or deficit respect to the evaporation demand (negative) (C_{RO}) and water flown through the bottom of the column (C_{BF}) during the whole period of time ($\eta = 20 \cdot \eta_0$).

- (a) profili di contenuto volumetrico d'acqua calcolati dalla procedura ai tempi adimensionali (multipli di η_0) indicati in figura; le soluzioni durante le fasi controllate dall'atmosfera sono di colore blu e rosso, durante le fasi controllate dal suolo sono di colore ciano e rosa; (b) flusso applicato alla superficie (linea tratteggiata spessa) e flusso analitico alla superficie dopo la saturazione (ciano) o il completo essiccamento (rosa), le due linee verticali (tratto punto) indicano i tempi di raggiunta saturazione (blu) e completo essiccamento alla superficie (rosso); (c) flussi cumulativi calcolati dalla procedura: flusso applicato alla superficie (C_{AF}), acqua entrata attraverso la superficie (C_{SF}), acqua guadagnata dalla colonna di suolo di altezza adimensionale $\zeta = 3$ (C_{WG}), ruscellamento (positivo) o deficit rispetto alla domanda di evaporazione (negativo) (C_{RO}) e flusso uscente dal fondo della colonna (C_{BF}) durante l'intero intervallo temporale ($\eta = 20 \cdot \eta_0$).

tial rate, that is, till the air-dry value is reached at the surface. At the desiccation time η_d starts a new soil-controlled phase. After $\eta = 11 \cdot \eta_0$ the soil water content profiles show a maximum, moving downward with decreasing amplitude. The thick dashed line in figure 1b corresponds to the boundary condition (flux applied at the surface). The first four cyan circles are the analytically computed surface flux values at the solution times, during the saturation; similarly, the last four circles (pink) are the surface flux values, during the desiccation. They are computed at the dimensionless times reported in figure 1a. The solid lines (cyan and pink) are the analytical surface flux computed each $\eta_0/5$ dimensionless time interval during the two soil-controlled phases. The two vertical dash-dot lines indicate the times of ponding η_p (blue) and air-dry soil η_d (red). Figure 1c reports the cu-

mulative fluxes obtained by the sum of the instantaneous flux, at the solution time, times the time interval between two consecutive solutions. The symbols indicate the values estimated at the solution times reported in figure 1a while the solid lines are obtained from the solutions computed each $\eta_0/5$ dimensionless time interval, as for figure 1b. In figure 1c a small difference between symbols and lines can be appreciated; it is due to the time interval between two consecutive solutions. The time interval for the symbols is η_0 or $2 \cdot \eta_0$ while, for the solid lines, it is $\eta_0/5$.

The computed runoff is the difference between the applied flux (boundary condition) and the analytically computed surface flux; therefore it is a real runoff when it is positive and it is the deficit respect to the evaporation demand when negative.

4.3. – HALF SPACE DOMAIN: NULL IC AND EXPONENTIAL FLUX BC

In this second application, the comparison between the soil water content obtained by the procedure and the results of the analytical solution of the linearized Richards equation subject to a null soil water content initial condition and a decreasing exponential flux boundary condition is presented. Therefore the initial-boundary conditions are:

$$\begin{aligned} \vartheta_i(\zeta) &= 0 & (\eta = 0) \\ \Phi_0(\eta) &= q_1 \cdot e^{-\gamma \eta} & (\zeta = 0) \end{aligned}$$

In this case the solution of the Richards equation is equal to equation (18a) or (18b) or (18c), according to the value of γ ; in the obtained solution $\Phi(\zeta, \eta)$, q_1 substitutes ϑ_i . Then the soil water content $\vartheta(\zeta, \eta)$ is obtained using equation (11):

$$\begin{aligned} \vartheta(\zeta, \eta) &= \frac{q_1}{V} e^{\left(\frac{\zeta-2\gamma\eta}{2}\right)} \\ &\left[\frac{-1}{2\gamma} e^{\left(\frac{\zeta+2\gamma\eta}{2}\right)} \operatorname{erfc}\left(\frac{\zeta+\eta}{2\sqrt{\eta}}\right) + \frac{e^{\frac{\tau\zeta}{2}}}{1-\tau} \operatorname{erfc}\left(\frac{\zeta+\tau\eta}{2\sqrt{\eta}}\right) + \frac{e^{-\frac{\tau\zeta}{2}}}{1+\tau} \operatorname{erfc}\left(\frac{\zeta-\tau\eta}{2\sqrt{\eta}}\right) \right] \end{aligned} \quad (19)$$

where $\tau = \sqrt{1-4\gamma}$. Equation (19) is valid for $\gamma \leq 1/4$ and till the saturation is reached.

In this second application of the procedure $\gamma = 1/4$ has been chosen, so that $\tau = 0$ and equation (19) is very simple. The soil water content distribution is computed for a column of soil of dimensionless height $\zeta = 3$, at multiples of the dimensionless time η_0 .

The results are shown in figures 2a, 2b and 2c.

The thick vertical dashed line on the left side of figure 2a is the null initial condition and the decreasing exponential in figure 2b is the boundary condition. The green and cyan solid lines in figure 2a are the solutions of the procedure obtained approximating the exponential boundary condition with a time step of $\eta_0/6$. The symbols represent the analytic soil water content profiles obtained using (19) with $\gamma = 1/4$. Both procedure and analytic solution are computed at the dimensionless times reported in figure 2a. Up to $\eta = 6 \cdot \eta_0$ the applied flux is able to increase the surface moisture without reaching the saturation, as shown by the first four solutions (green curves). As the time goes on, the decreasing exponential flux implies a decrease in the surface moisture, as shown by the last four solutions (cyan curves). In order to qualitatively verify the influence of the “time step” chosen to approximate the boundary condition, the solutions at 1, 2, 4 and 6 η_0 have been recomputed using a time step of 4 η_0 ; they are the black lines in figure 2a. In figure 2c only the cumulative applied surface

flux (black), the water gained by the column (blue) and the cumulative bottom flux (green) are reported because the surface flux equals at each time the applied one and, therefore, the runoff is always null.

4.4. – FINITE-THICKNESS DOMAIN: CONSTANT IC AND BCs

A simple example, with constant initial-boundary conditions for the finite-thickness domain, is here presented. In this case the solution of the model is exactly the analytical solution of the problem. In fact, the solution is the sum of equations (12a) and (12b) which are the basic elements to build a procedure similar to the one described for the half-space domain. Uniform soil water content at saturation ($\vartheta = 1$) is assumed as initial condition for the finite-thickness layer of dimensionless depth $\zeta = 1$. At the time origin the soil water content at the top of the layer becomes and remains null while the value at the bottom continues to be at saturation. Figure 3a shows the soil water content profiles computed by equations (12a) and (12b) at the dimensionless times: 1, 2, 4, 6, 12, 24, 36, 48, 72, 96, 120, 144, 192, 240 η_0 . The dashed line is the initial condition and the dotted line is the stationary solution. Figure 3b shows the instantaneous dimensionless flux at the top (pink) and bottom (green) boundaries; the symbols are the values computed at the aforementioned solution times. The fluxes are positive in the increasing depth direction. The square crossed is the value to which both the fluxes tend: that is the stationary flux. Figure 3c shows the cumulative dimensionless flux at the top (red) and bottom (green) boundaries. The total water content trend (blue line and star) is also shown. The full blue star is the total water content at the stationary condition.

5. – CONCLUSIONS

In this paper, a compilation of the solutions obtained for the linearized one-dimensional Richards equation, solved both in a half-space and in a finite-thickness domain has been presented. For space reasons, only the most meaningful solutions were described.

The soil water content at any required time and depth results as the sum of two components; one related to the initial condition and to null boundary conditions and the other related to the boundary conditions and to null initial condition. The sum of these two components is the general solution of the Richards equation in integral form and the

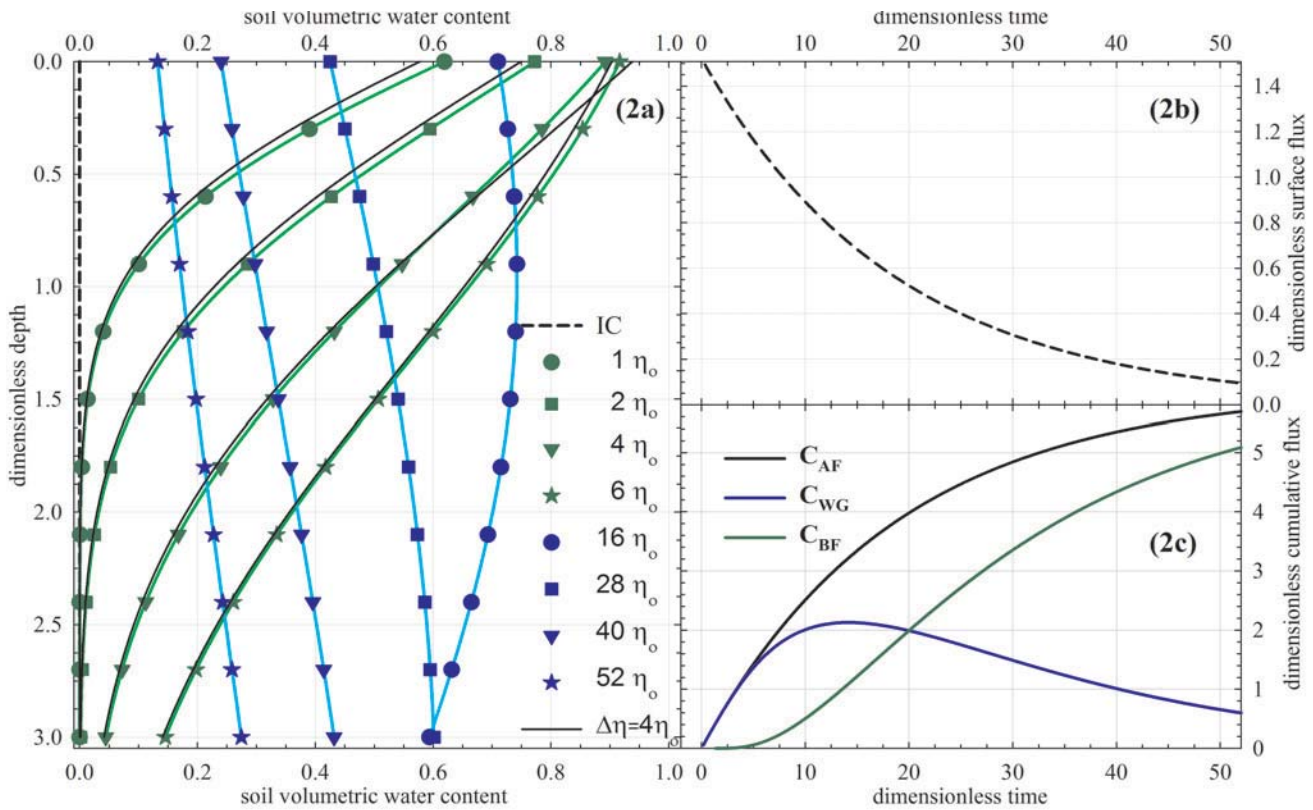


Fig. 2 - (a) soil moisture profiles computed by the procedure (solid lines) and using equation (19) (symbols) at the indicated dimensionless times (multiples of η_0). The four black lines are the solutions computed by the procedure (at the same times of the green curves) using a worst approximation of the boundary condition; (b) exponential flux applied at the surface; (c) cumulative trends computed by the procedure: applied surface flux (C_{AF}), water gained by the column of soil of dimensionless height $\xi=3$ (C_{WG}) and water flown through the bottom of the column (C_{BF}) during the whole period of time.
 - (a) profili di contenuto volumetrico d'acqua calcolati dalla procedura (linee continue) e con l'equazione (19) (simboli) ai tempi adimensionali (multipli di η_0) indicati in figura. Le quattro linee nere sono le soluzioni, calcolate agli stessi tempi di quelle verdi, ma con un'approssimazione della condizione al contorno meno accurata; (b) flusso esponenziale applicato alla superficie; (c) andamenti cumulativi calcolati dalla procedura: flusso applicato alla superficie (C_{AF}), acqua guadagnata dalla colonna di suolo di altezza adimensionale $\xi=3$ (C_{WG}) e flusso uscente dal fondo della colonna (C_{BF}) durante l'intero intervallo temporale.

analytical expression of the soil water content distribution is therefore obtained if the integrals in the solution can be solved. The paper also describes a new method to compute the solutions of the linearized Richards equation, when arbitrary discrete initial and boundary conditions are available (step functions). For the half-space domain the initial condition is the soil volumetric water content and the boundary condition is the flux available at the surface. The experimental readings at a hydro-meteorological station correspond exactly to a step function like the one required by the model. This model also accounts for the switching between successive atmosphere-controlled and soil controlled phases and vice versa. Each phase has its own solution and requires its own initial condition, which is automatically computed by the described procedure. If the linearization of the Richards equation can be considered a valid assumption for a specific soil and its soil water content, the presented procedure is a quite valid tool to estimate the evolution of the soil water content

distribution, the ponding or desiccation time, the water fluxes and the water gained or lost by a column of surface soil. It can also be used, a priori, to define the best spatial domain and the probes depth of a hydrological station to study a specific topic: irrigation, evaporation, ground water recharge and so on. The same can be said about the time step of meteorological readings to be used to estimate the flux applied at the surface. It must be remarked that the spatial domain of the procedure is a uniform half-space. In many real cases (e.g. in presence of shallow freatic aquifers) the spatial domain is finite and it is better represented by a finite-layer. Here a very simple example for the finite-thickness domain has been presented. The importance of this example is that its analytical solution, which is derived for constant initial and boundary conditions, represents the basic element to build a new procedure similar to the one described for the half-space domain. In fact, the solution with the initial-boundary conditions approximated with step functions is a sum of

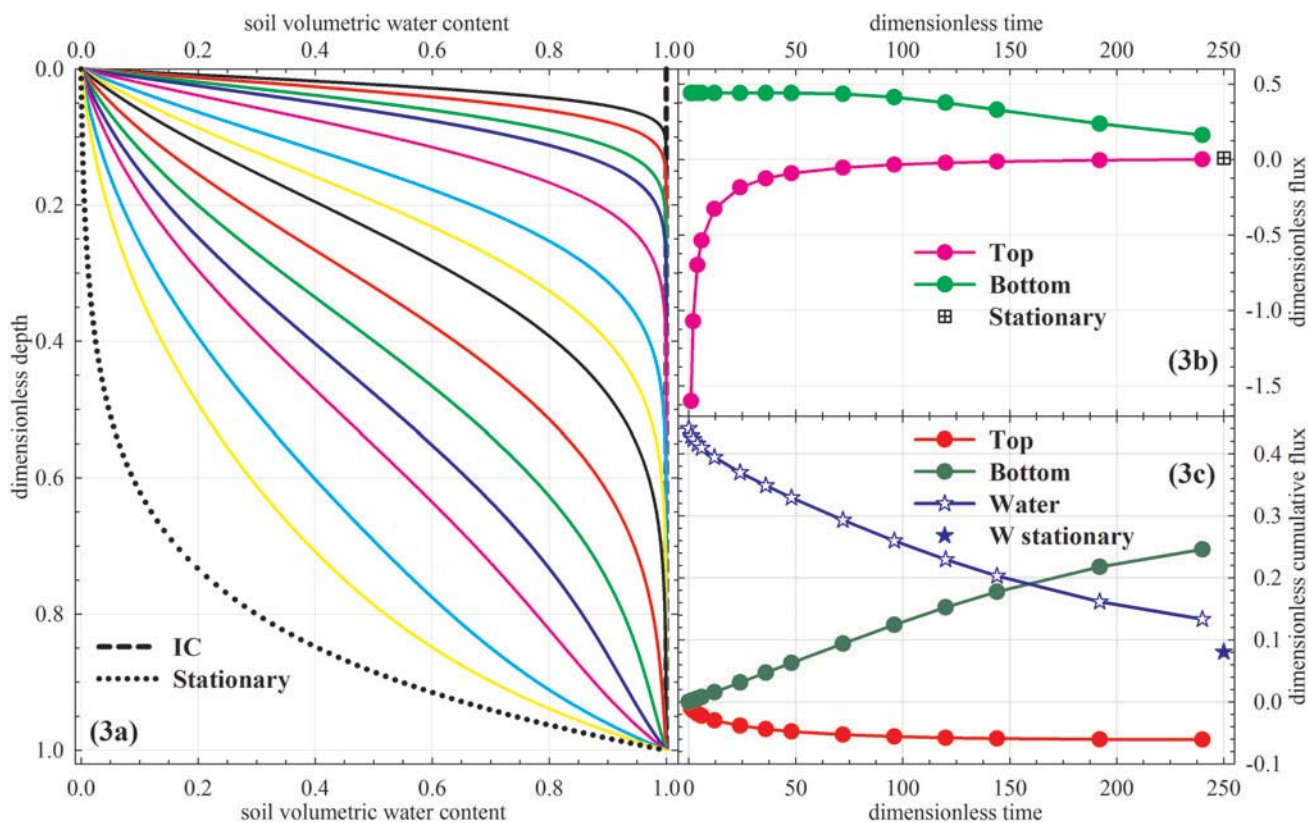


Fig. 3 - (a) soil moisture profiles computed by equations (12a) and (12b) at the dimensionless times: 1, 2, 4, 6, 12, 24, 36, 48, 72, 96, 120, 144, 192, 240 η_0 . The dashed line is the initial condition ($\theta = 1$); the dotted line is the stationary solution; (b) instantaneous dimensionless flux at the top (pink) and bottom (green) boundaries; the symbols are the values computed at the aforementioned solution times. The square crossed is the value of the flux at the stationary condition; (c) cumulative dimensionless flux at the top (red) and bottom (green) boundaries, the symbols are the values computed at the solution times; total water content trend (blue), the symbols (star) are the values computed at the solution times; the full blue star is the total water content at the stationary condition.

- (a) profili di contenuto volumetrico d'acqua calcolati dalle equazioni (12a) e (12b) ai tempi adimensionali: 1, 2, 4, 6, 12, 24, 36, 48, 72, 96, 120, 144, 192, 240 η_0 . La linea tratteggiata è la condizione iniziale ($\theta = 1$); la linea punteggiata è la soluzione stazionaria; (b) flussi istantanei adimensionali ai confini superiore (rosa) e inferiore (verde); i simboli sono i valori calcolati ai tempi delle soluzioni citati sopra. Il quadrato con la croce all'interno è il valore del flusso in condizione stazionaria; (c) flussi cumulativi adimensionali ai confini superiore (rosso) e inferiore (verde), i simboli sono i valori calcolati ai tempi delle soluzioni la stella blu piena indica il contenuto d'acqua alla stazionarietà.

solutions similar to the one here used. In this case (finite-thickness domain), since the boundary conditions are the soil water content trends, the procedure doesn't require a switch between atmosphere-controlled and soil-controlled phases.

Acknowledgments

The work here presented is the theoretical part of the models developed by the authors in the framework of National Italian Project: "Esplorazione geofisica e geologica di alcuni complessi acquiferi alluvionali nella pianura Padana tra Milano e Bologna, per la modellazione della circolazione idrica sotterranea" (PRIN 2005043388) and also during the first year of the new one: "Studio integrato geofisico, geologico, petrofisico e modellistico-matematico di complessi acquiferi alluvionali rappresentativi del sottosuolo padano: relazioni tra scala della ricostruzione idrostratigrafica e modelli di flusso" (PRIN 2007F9M9L8). These models will be applied to the experimental data measured at three stations located between the town of Parma and the Taro River. The authors wish to thank "Ciclo Idrico Integrato del-

l'Enia Parma" and "Barilla" for the availability of the places where the instruments are installed.

The Italian Ministero dell'Istruzione, dell'Università e della Ricerca; the Università di Modena e Reggio Emilia; the Fondazione Cassa di Risparmio di Modena in part funded this work.

The authors wish to thank Prof. Mauro GIUDICI and Dr. Emanuele ROMANO for the useful suggestions and revision of the manuscript.

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